Chapter Zero - Math Skills

0.1 Symbolic Manipulation

Algebra includes symbolic manipulation of variables and the use of specialized techniques such as factoring, the Zero-Product Rule and the Quadratic Formula. One goal is to be able to solve an equation for one of its variables.

Example 0.1 Solving an Equation for a Variable

If $v = 10 + 5t$, then what is t?

This equation has two variables: v (the object's final speed) and t (the elapsed time) In this case, we are asked to determine the elapsed time. If we subtract 10 from both sides and divide both sides by 5, then the variable t will be all by itself on the left-hand side of the equation. The result is

$$t = \frac{v - 10}{5}$$

We can also solve an equation which is comprised of only symbols for one of its variables. In the above equation, the number 5 stood for the rate at which the object was accelerating (5 meters per second squared) and 10 stood for the object's initial speed (10 meters per second). If neither of these were given, then the equation would be

$$v = v_o + at$$

This equation has four variables: v (the object's final speed), v_o (the object's initial speed), a (the rate at which the object was accelerating) and t (the elapsed time). If we subtract v_o from both sides and then divide both sides by a, then the variable t will be all by itself on the left-hand side of the equation and the result would be

$$t = \frac{v - v_o}{a}$$

Therefore, as you can see, it doesn't matter whether we are dealing with numbers or symbols. The rules of algebra are always the same.

Now You Try It Exercise 0.1: If $c^2 = a^2 + b^2$ and a is greater than zero, then what is a?

(A) $c - b$ (B) $c^2 - b^2$ (C) $b^2 - c^2$ (D) $\sqrt{c^2 - b^2}$ (E) $\sqrt{b^2 - c^2}$

0.2 Factoring and the Zero-Product Rule

In many cases you can solve an equation by factoring and using the **Zero-Product Rule:**

If the product of any number of factors is zero, then one of the factors must be zero.

Example 0.2 Factoring and using the Zero-Product Rule

If $x^3 = 3x^2$, then what is x?

Your first instinct might be to divide both sides by x^2 and arrive at $x = 3$. This answer is correct, but it is not complete. The better approach is to subtract $3x^2$ from both sides, factor and use the Zero-Product Rule.

$(x^2)(x - 3) = 0$

The Zero-Product Rule states that in this case either $x^2 = 0$ or $x - 3 = 0$. Solving these two equations yields the complete solution set of $x = 0$ or $x = 3$.

Now You Try It Exercise 0.2: If $x^4 = 9x^2$, then what is x?

(A) 0 or 3 (B) 3 or -3 (C) 0, 3 or -3 (D) 0, 3 or 9 (E) 0, 9 or -9

0.3 The Quadratic Formula

A special case occurs when the largest exponent of a polynomial equation is two. In this case, the equation is called a quadratic equation and it can be written in the form $ax^2 + bx + c = 0$. The value or values of x can be determined by using the **Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (0.1)$$

Example 0.3 Using the Quadratic Formula

If $5x^2 - 5x = 30$, then what is x?

If we subtract 30 from both sides and divide both sides by 5, then:

$x^2 - x - 6 = 0$

This is a quadratic equation where $a = 1$, $b = -1$ and $c = -6$.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)}$$

$$x = \frac{1 \pm 5}{2} = 3 \text{ or } -2$$

Alternatively, we could factor the equation and use the Zero-Product Rule.

$(x - 3)(x + 2) = 0$

In this case, either $x - 3 = 0$ or $x + 2 = 0$ which yield $x = 3$ or $x = -2$. Notice that each method produces the same two solutions.

Now You Try It Exercise 0.3: If $x^3 - 6x^2 + 8x = 0$, then what is x?

(A) 2 or 4 (B) 0, 2 or 4 (C) -2 or -4 (D) 0, -2 or -4 (E) 2, -2, 4 or -4

0.4 Geometry

Geometry includes methods to determine the lengths of curved lines, and the surface areas, cross-sectional areas and volumes of three-dimensional shapes.

The length of a curved line that is a piece of a circle is found by using the **arc length formula**.

$$s = r\theta \tag{0.2}$$

where s is the length of the curved line, r is the radius of the circle of which the line is a piece and θ is the angle measured in radians between the two radii that connect the two ends of the curved line to the center of the circle. (See Figure 0.1.)

(insert Figure 0.1)

To check this relationship, we could determine the length of the entire circle, which is called the **circumference**. If you go around the circle once, then the angle is 2π radians.

$$s = r\theta = (r)(2\pi) = 2\pi r = C = \text{the circumference of the circle.}$$

This is confirmation that the arc length formula is correct.

The **surface area** of an object is the area of the boundary between the object and its environment. For example, a cube is a shape where the length, width and height are all equal and where all of the sides meet each other at 90° angles. On its surface, a cube has six squares which all have the same size. The surface area of a cube is therefore equal to six times the area of one of the squares, or $A = 6L^2$.

The **cross-sectional area** of an object is the area that is created when the object is intersected by a single, flat plane. Imagine slicing through an object with a knife in one direction. One side of the interior of the object that is exposed along the cut would be the cross-sectional area. For example, a cylinder is a shape that has a constant circular cross-sectional area perpendicular to its height. The cross-sectional area of a cylinder perpendicular to its height is therefore the area of a circle, which is equal to π times the radius of the circle squared, or $A = \pi R^2$.

(Insert Illustration of a Chef Slicing a Roast)

The **volume** of an object is the amount of three-dimensional space that it occupies. For example, the volume of a cube is the length of one of its sides cubed, or $V = L^3$, and the volume of a cylinder is its perpendicular cross-sectional area times its height, or $V = \pi r^2 h$.

Other surface areas, cross-sectional areas and volumes are listed in Appendix E.

Now You Try It Exercise 0.4: What is the volume of a sphere?

(A) $2\pi r$ (B) $4\pi r^2$ (C) $4\pi r^3$ (D) $\frac{4}{3}\pi r^2$ (E) $\frac{4}{3}\pi r^3$

0.5 Trigonometry

Trigonometry is the geometry of triangles. You'll be using trigonometry frequently. Here is a review of the important relationships.

The Pythagorean Theorem is the relationship between the lengths of the three sides of a right triangle. A right triangle is a triangle that has a 90° angle in it. The longest side of the triangle is called the **hypotenuse**. The Pythagorean Theorem says that the square of the hypotenuse is equal to the sum of the squares of the other two sides. (See Figure 0.2.) Symbolically,

$$c^2 = a^2 + b^2 \qquad (0.3)$$

(Insert Figure 0.2)

Trigonometric functions relate two sides of a triangle and one of the angles. There are six trigonometric functions: *sine, cosine, tangent, secant, cosecant* and *cotangent*. The definitions of these functions are listed in Appendix E.

The Law of Cosines is a generalization of the Pythagorean Theorem for any triangle, not just for right triangles. It looks similar to the Pythagorean Theorem, with an extra term.

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad (0.4)$$

Angle C is opposite side c. (See Figure 0.3.) Note that if angle C is 90°, then $\cos C$ is zero and the Law of Cosines reduces to the Pythagorean Theorem.

(Insert Figure 0.3)

The Law of Sines gives ratios that are the same for any triangle.

$$(\sin A)/a = (\sin B)/b = (\sin C)/c \qquad (0.5)$$

The angles A, B and C are opposite sides a, b and c, respectively.

Additionally, the three interior angles of any triangle always add up to 180°.

$$A + B + C = 180° \qquad (0.6)$$

Now You Try It Exercise 0.5: If $\sin\theta = 0.6$, then what is $\tan\theta$?

(A) 0.75 or -0.75 (B) 0.8 or -0.8 (C) 1.25 or -1.25 (D) 1.33 or -1.33 (E) 1.67 or -1.67

0.6 Equations, Symbols and Units

Equations in math are like sentences in English. They communicate fundamental mathematical ideas in a written form. When you write solutions to physics problems, you will almost always be writing in the language of mathematics. Be sure to use complete equations when you write.

A complete equation has three parts: 1) the left-hand side (LHS), 2) an equal sign (=) and 3) the right-hand side (RHS). Whenever you write an equation, you should make sure that the two things on either side of the equals sign are in fact the same.

In physics, the answer to a problem will usually be an equation. The following format should be used. Write the **symbol** of the quantity on the LHS and write what it is equal to on the RHS. In pure math, what is on the RHS is just a number. In physics, however, the number must almost always be accompanied by one or more **units**. In most cases, a number without a unit is completely meaningless. Here is an example:

Example 0.4 The Importance of Units

(Insert Street Map of a Cambridge, MA)

If you ask me how far my house is from campus and I answer "6", then have I really told you anything? No, because I have not specified the units. By specifying the units along with the number I will have nailed down the distance to a specific value. Here are some possibilities:

x = 6 miles
x = 6 kilometers
x = 6 blocks

Therefore, in most cases, a number without units is completely meaningless.

Most rules have exceptions, and this rule is no different. Some quantities have no units. A quantity without units is very rare. These rare cases will be noted as they arise.

Quantities are far superior to units for the following reason. Quantities such as distance, time, speed, force, and energy are determined by the physical structure of the universe in which we live. Units such as feet, miles, seconds, minutes, miles per hour, pounds and ounces are completely arbitrary human constructs. Therefore, when discussing or writing about a particular event or condition, refer to the quantities, not the units. For example, ask, "What is the distance?" instead of "What are the miles?" *Units are arbitrary. Quantities are not.*

You'll be using SI units. SI stands for International System, or Systeme Internationale, in French. Most people refer to this system as the metric system. A useful abbreviation for this system of units is **MKS**, which stands for meters, kilograms and seconds. This abbreviation will be most useful because it will help us remember the units for distance, mass, and time. Here is a list of the base units we will be using, along with their corresponding quantities in the MKS System.

Base MKS Units	**Quantity**
meters	distance
kilograms	mass
seconds	time
amperes	electrical current
radians	angle
kelvin	temperature
mole	amount of substance

More complicated units can be formed by multiplying and/or dividing base units. For example, the MKS unit of force, the newton, is equal to kilograms times meters divided by seconds squared, or in abbreviated form N = kg m/sec^2.

When writing very large or very small numbers, it is often useful to use a prefix. **Prefixes** are letters that stand for large or small numbers and are used in order to write less. *The most important thing to remember is that even though prefixes are letters they represent numbers and not units.* A list of prefixes is on the inside front cover.

Another way to write less is to use **Scientific Notation**. In this notation, a *non-zero* digit is written to the left of the decimal point, one or more digits are written to the right of the decimal point and this number is multiplied by ten raised to some integer exponent.

Example 0.5 Shorthand Notation

Suppose that the strength of a very small force is determined to be F = 0.00512 N. This value can be written using fewer symbols in two ways. (As you will see, physicists and mathematicians like to write as little as possible.)

(Insert Caricature Drawing of Albert Einstein)

a. One way would be to use Scientific Notation. In this form, the strength of the force would be $F = 5.12 \times 10^{-3}$ N.

b. Another way would be to use a prefix. The prefix m stands for milli which means one one-thousandth or 10^{-3}. Using a prefix, F = 5.12 mN.

Now You Try It Exercise 0.6: What number corresponds to the prefix M = mega?

(A) 10^{-6} (B) 10^{-3} (C) 10^{3} (D) 10^{6} (E) 10^{9}

0.7 Unit Analysis

There are two different types of equations: those that relate quantities and those that relate units. Here is an example. Speed is equal to distance divided by elapsed time. The symbols for speed, distance and elapsed time are, respectively, v, Δx and Δt. The MKS units of speed, distance, and time are, respectively, meters per second, meters and seconds.

v	=	$\Delta x/\Delta t$	(Quantity Equation)
(m/sec)	=	(m)/(sec)	(Unit Equation)

In this case, the unit equation can be analyzed to check the quantity equation. But this is not always true. Here is another example. The area of a circle is equal to π times the radius of the circle squared. Imagine now that you make a mistake and forget the π. You write the area is equal to the radius squared. The quantity equation is wrong. However, the unit equation is still correct because π has no units. It is just a dimensionless number.

In the past you may have learned to check an equation by looking at the corresponding unit equation. This procedure is called **unit analysis**. This method is good, but there are better ways to go about checking your work. *Unit analysis can only tell you if a quantity equation is incorrect. It cannot tell you if it is correct.* Therefore, unit analysis will is not stressed. Instead, we will focus on quantities.

Now You Try It Exercise 0.7: Which of the following is *incorrect* by unit analysis?

(A) $1/v = \Delta t/\Delta x$ (B) $\Delta x = v/\Delta t$ (C) $\Delta t = \Delta x/v$ (D) $\Delta x = v\Delta t$

0.8 Significant Figures

The number of **significant figures** is related to the precision of a measurement or given value. The **1st significant figure** is the one that is furthest to the left that is not zero.

Quantity and Numeric Value	**1st Significant Figure**
x = 51,234 meters	5
x = 0.051234 meters	5
x = 3,074,000 meters	3

Notice that the location of the decimal point has nothing to do with significant figures.

Example 0.6 Rounding

Answers are rounded to the appropriate number of significant figures. In the following example, the values are rounded to three significant figures.

Value Before Rounding	Value After Rounding
t = 0.0000456889 sec	t = 0.0000457 sec
t = 123,400 sec	t = 123,000 sec
t = 0.101101 sec	t = 0.101 sec

Notice again that the location of the decimal point has nothing to do with significant figures.

If the quantity is an integer, then the answer would never have a decimal point or any digits smaller than the ones digit. For example, if we determine that it takes ten people to lift a piano up off the ground, then we would write the answer as N = 10 people and not N = 10.0 people because it is absurd to refer to a non-integer number of people.

Now You Try It Exercise 0.8: How many significant figures does 0.0010101 have?

(A) 3 (B) 4 (C) 5 (D) 7 (E) 8

0.9 Solving Simultaneous Equations

When solving physics problems, it is very rare to simply take a formula, put in numbers and get another number. Typically, two or more relationships are considered. You will need to be able to combine the information in multiple relationships. Combining two or more relationships to get answers is called **solving simultaneous equations**, and it can be accomplished in three ways: *substitution, multiplication and subtraction*, or *forming a ratio*. Some techniques are easier than others, and some are more general. With practice, you will recognize which technique will be most useful in each situation. The three techniques are described in detail below.

Substitution is by far the most difficult and tedious method of the three. However, it is the most general, that is, it always works. The other two methods only work in certain cases. When in doubt, use the substitution method because it always works. Here are the steps:

1) Identify which equation is simplest.
2) Solve for one of the variables in the simplest equation.
3) Substitute the expression for that variable into one of the other equations.
4) Repeat if necessary until only one variable remains.
5) Solve the equation for the remaining variable using the rules of algebra.
6) Solve for the other variables.
7) Check the answers.

Example 0.7 Using the Substitution Method

We have already determined that the following two relationships are important in determining the values of x and y. Now we want to combine the two relationships in order to determine the values of x and y that make both equations true.

$3x + 2y = 7$ and $5x - y = 3$

1) The equation on the right is simpler because y is not being multiplied by a number.
2) Solve for y in the right equation: $y = 5x - 3$
3) Substitute this expression into the other equation: $3x + 2(5x - 3) = 7$
4) Since there is only one variable left (x) we do not need to repeat any steps.
5) Solve for x: $3x + 10x - 6 = 7$
   $13x = 13$
   $x = 1$
6) Solve for y using $y = 5x - 3$: $y = 5(1) - 3 = 2$
7) Check the answers. The right equation yields 7 = 7 and the left one yields 3 = 3 when the values of x and y are used. The answers $x = 1$ and $y = 2$ are correct.

Now You Try It Exercise 0.9: $x^2 + y^2 = 13$ and $x - y = 1$. What are x and y?

(A) $x = 3$, $y = 2$ or $x = -2$, $y = -3$ (B) $x = 3$, $y = 2$ (C) $x = -2$, $y = -3$

The **Multiply and Subtract** method is much easier. However, it only works in certain cases.

Example 0.8 Using the Multiply and Subtract Method

We'll use the same equations as in Example 0.7 to demonstrate this method. When we multiply the second equation by -2 and subtract it from the first equation, the y values will cancel.

$$
\begin{array}{llll}
 & & (3x + 2y & = 7) \\
- & -2 & (5x - y & = 3) \\
\hline
 & & 13x & = 13
\end{array}
$$

We arrive at the same equation as in step 5 of the substitution method above, but with many fewer steps. However, if y is replaced with y^2 and x is replaced with x^2 in one of the equations, then we can't use multiply and subtract because no matter what number we multiply by, when we subtract there won't be any cancellation.

Forming a ratio is the easiest technique of the three. However, it only works when there are *two* equations that involve only multiplication and/or division. Here is an example:

Example 0.9 Using the Ratio Method

In thermal radiation, the power output can be found by the following equation:

$$P = e\sigma AT^4$$

where P is the power output, e is the emissivity, σ is a constant, A is the surface area of the radiating object and T is the temperature. Initially, the power output is 100 W and the temperature is 100 K. If the temperature is increased to 200 K and the emissivity and surface area remain the same, then what will be the new power output?

If we use a subscript of 1 to indicate the initial case, and a subscript of 2 to indicate the final case, and divide the two equations by each other to form a ratio, we get:

$$\frac{P_2 = e\sigma AT_2^4}{P_1 = e\sigma AT_1^4} \qquad\qquad \frac{P_2}{P_1} = \frac{T_2^4}{T_1^4}$$

Now, this equation has only one unknown (P_2). The result is $P_2 = 1600$ W. Notice that we determined the final power output without ever knowing the emissivity or the surface area of the radiating object. Pretty cool, huh?

0.10 Scalar and Vector Quantities

Unless you have taken physics before, you have probably never heard of scalar or vector quantities. These terms refer to the two fundamental types of mathematical objects that we will be using. Whenever a new physical quantity is encountered, it will be important for you to know whether it is a scalar or a vector quantity.

A **scalar** quantity is simply a number. The number can be positive or negative. At this level, scalar quantities will always be real numbers, never imaginary or complex numbers. If you find yourself taking the square root of a negative number, then you know that you have made a mistake because the result would be an imaginary or complex number.

A **vector** quantity is comprised of two things: a number and a direction. The technical term for the number is **magnitude**. Other terms for the number are size and amount. The number can also be thought of as the length of the vector. However, not all vectors describe lengths so you must be very careful when thinking about a vector in terms of its length.

Almost all of the fundamental quantities in physics are vectors, not scalars. The exceptions are mass, time, energy, number of moles, electrical current and quantities that are either derived from them or are the result of combining them. Power, for example, is a combination of energy and time. Therefore, power is also a scalar. Temperature is related to energy, so temperature is also a scalar.

Here is an example to help you understand the difference between vectors and scalars.

Example 0.10 Speed and Velocity

(Insert Illustration of a Freeway with Cars Traveling in both Directions)

Imagine driving from Los Angeles to San Francisco on the freeway. Speed is a scalar that would tell you how fast you are driving. The symbol for speed is v. If you were driving at 70 MPH, then we would write v = 70 MPH. On the other hand, velocity is a vector which would tell you how fast and also in which direction. The symbol for velocity is the same as speed, except it has an arrow above it to indicate that it is a vector. In this example, your velocity would be written as $\vec{v}$ = (70 MPH, north). If you were driving in the other direction, from San Francisco to Los Angeles, then your speed would be the same, but your velocity would be different. Your velocity now would be $\vec{v}$ = (70 MPH, south).

We are concerned about direction because it will usually make a big difference. Imagine two people pushing equally hard on a heavy object that is initially not moving. If they push in the same direction, then the object might move, if they push hard enough. But, if they push in opposite directions, then, no matter how hard they push, the object will not move because the effects of their pushing will always cancel.

It is relatively easy to add and subtract scalars, and relatively difficult to multiply and divide them. For example, 113 + 23 is easy; you can do it in your head, right? 113 x 23 is more difficult. You probably need a piece of paper and a pen or pencil to get the right answer.

Vectors are the exact opposite. It is relatively easy to multiply and divide vectors, and relatively difficult to add and subtract them. The techniques for adding and subtracting vectors will be outlined in the next two sections.

Vector Addition and Subtraction: Graphical Method

In practice, we will never add vectors graphically because the technique is not accurate enough for our needs, unless we use some fancy drafting software, which most of us don't have. However, it will often be a good idea to get a rough idea using a pen or a pencil so that when we are finished adding or subtracting the vectors algebraically we can check our answer.

A vector will be drawn as an arrow (a straight line with an arrowhead at one end only). The end without the arrowhead is called the tail. The end with the arrowhead is called the tip. This arrow points in the direction of the vector.

To add two vectors graphically, follow these steps. This is called the **tail-to-tip method**. (See Figure 0.4.)

1) Draw a set of coordinate axes (horizontal and vertical lines).
2) Draw the first vector ($\vec{A}$) with its tail at the origin.
3) Draw the second vector ($\vec{B}$) starting at the tip of the first vector.
4) Draw a straight line connecting the origin to the tip of the second vector.
5) The third line ($\vec{C}$) is the sum of the two vectors.

(Insert Figure 0.4)

To subtract one vector from another, multiply the vector subtracted ($\vec{B}$) by -1. Now, add the first vector and the reversed second vector using the steps above. The result will be the difference of the two vectors. (See Figure 0.5.)

(Insert Figure 0.5)

In the next section we will learn how to add and subtract vectors using algebra. In practice, we will always add and subtract them this way.

Vector Addition and Subtraction: Component Method

There are two ways to describe a vector. The first way is called **polar form**. This is the way that the vectors will usually be given. It is also the way that the answer will usually be written. *Polar form gives the magnitude and the direction*. The direction is usually measured by starting on the horizontal line pointing to the right (the positive x-axis) and then rotating counter-clockwise until you reach the vector. This is called measuring the angle in **standard polar form**.

The other way to describe a vector is called Cartesian form, named after Rene Descartes, the famous mathematician and philosopher. **Cartesian form** is used to add and subtract vectors and it describes how much the vector points horizontally and vertically. These two aspects of a vector are called components. *Components that are to the right or up will be positive, while those to the left or down will be negative.* Here are two examples.

Example 0.11 Converting from Polar to Cartesian Form

(Insert Figure 0.6)

Imagine beginning a journey in New York City. You are going to fly along a straight line to Boston, which is 100 miles northeast from New York City. (Northeast is exactly halfway between north and east.) (See Figure 0.6.) In polar form, your displacement from New York City to Boston would be $\Delta\vec{R}$ = (100 miles, 45.0°). What would your displacement be in Cartesian form? If we drew a diagram, made a right triangle and used the definitions of sine and cosine, then we could determine the distance east and the distance south:

$$\Delta\vec{x} = \Delta R\cos\theta = 100 \cos 45.0° = +70.7 \text{ miles}$$
$$\Delta\vec{y} = \Delta R\sin\theta = 100 \sin 45.0° = +70.7 \text{ miles}$$

The displacement in Cartesian form would therefore be $\Delta\vec{R}$ = (70.7 miles, 70.7 miles). The components are equal because sine and cosine are equal when the angle is 45.0°.

Example 0.12 Converting from Cartesian to Polar Form

(Insert Figure 0.7)

The process also works in reverse. Imagine that a destination is 100 miles east and 50.0 miles north. (See Figure 0.7.) In Cartesian form, $\Delta\vec{R}$ = (100 miles, 50.0 miles). What would be the displacement in polar form? Again, if we drew the diagram and used trigonometry, then:

$$\Delta R = (\Delta x^2 + \Delta y^2)^{1/2} = (100^2 + 50.0^2)^{1/2} = 112 \text{ miles}$$
$$\theta = \tan^{-1}(\Delta y/\Delta x) = \tan^{-1}(50.0/100) = 26.6°$$

The displacement in polar form would therefore be $\Delta\vec{R}$ = (112 miles, 26.6°).

Polar-to-Cartesian Transformation Equations

$$\Delta\vec{x} = \Delta R \cos\theta \qquad (0.7)$$
$$\Delta\vec{y} = \Delta R \sin\theta \qquad (0.8)$$

Cartesian-to-Polar Transformation Equations

$$\Delta R = (\Delta x^2 + \Delta y^2)^{1/2} \qquad (0.9)$$
$$\theta = \tan^{-1}(\Delta y / \Delta x) \qquad (0.10)$$

Note: In order for equations 0.7, 0.8 and 0.10 to work correctly, the angle θ must be expressed in standard polar form.

Now that we know how to change the form of a vector, we are ready to learn how to add and subtract them algebraically.

To add two vectors algebraically, follow these steps. This is called the **Component Method**.

1) Use sine and cosine to compute the horizontal and vertical components of the vectors.
2) Add the two horizontal components together. Do the same for the vertical ones.
3) Convert back to polar form for the final answer.

To subtract two vectors algebraically, do the same steps above, except in step two subtract the components instead of adding them.

Example 0.13 Vector Addition using the Component Method

(Insert Figure 0.8)

If I walk northeast for 10.0 miles and then southeast for 5.00 miles, then how far am I from the starting point? (See Figure 0.8.) The answer is not 15.0 miles because I did not walk in the same direction the whole time. Following the steps above, I can determine the total distance.

$\Delta\vec{R}_1$ = (10.0 miles, 45.0°) and $\Delta\vec{R}_2$ = (5.00 miles, 315°)

where the angles are given in standard polar form.

$\Delta\vec{x}_1 = \Delta R_1 \cos\theta_1$ = 10.0 cos 45.0° = +7.07106 miles
$\Delta\vec{x}_2 = \Delta R_2 \cos\theta_2$ = 5.00 cos 315° = +3.53553 miles
----
$\Delta\vec{x}$ = 7.07106 + 3.53553 = 10.60659 miles

$\Delta\vec{y}_1 = \Delta R_1 \sin\theta_1$ = 10.0 sin 45.0° = +7.07106 miles
$\Delta\vec{y}_2 = \Delta R_2 \sin\theta_2$ = 5.00 sin 315° = - 3.53553 miles
----
$\Delta\vec{y}$ = 7.07106 + (-3.53553) = 3.53553 miles

$\Delta R = (\Delta x^2 + \Delta y^2)^{1/2}$ = $(10.60659^2 + 3.53553^2)^{1/2}$ = 11.2 miles

Notice that the horizontal components combine to make a larger total. This is because I walked east all the time, never west. Also notice that the vertical components partially cancel. This is because some of the time I walked north and some of the time I walked south.

Now You Try It Exercise 0.10: $\vec{A}$ = (5.00 meters, east) and $\vec{B}$ = (5.00 meters, 30.0° west of north). If $\vec{C} = \vec{A} + \vec{B}$, then what is $\vec{C}$ in polar form? (See Figure 0.9.)

(Insert Figure 0.9)

(A) (10.0 meters, 30.0° north of east) (B) (10.0 meters, 30.0° east of north)
(C) (5.00 meters, 30.0° east of north) (D) (5.00 meters, 30.0° east of south)

0.11 Exponents and Logarithms

An exponential equation is a compact way of writing a number multiplied by itself any number of times. The following notation is used.

base $^{\text{exponent}}$ = answer

The base is the number being multiplied by itself, the exponent is the number of times the base is multiplied and the answer is the result of the multiplication. For example, $5^3 = 5 \text{ x } 5 \text{ x } 5 = 125$. If the exponent is one, then the answer is equal to the base. For example $4^1 = 4$. If the exponent is zero, then the answer is one. For example, $6^0 = 1$. If the exponent is negative, then the answer is one divided by what the answer would be if the exponent was positive. For example, $5^{-2} = \frac{1}{5^2} = \frac{1}{25} = 0.04$.

The following properties of exponents can come in handy from time to time.

$A^x A^y = A^{x+y}$	(Add the exponents together.)	(0.11)
$\frac{A^x}{A^y} = A^{x-y}$	(Subtract one exponent from the other.)	(0.12)
$(A^x)^y = A^{xy}$	(Multiply the exponents together.)	(0.13)

A logarithmic equation is another way of expressing the same information contained in an exponential equation. Technically, exponential and logarithmic functions are inverse functions of each other. The following notation is used for logarithmic equations.

log $_{\text{base}}$ answer = exponent

It is often useful to rewrite logarithmic equations as exponential equations, and vice versa. For example, $\log_{10} 100 = 2$ is equivalent to $10^2 = 100$. If the answer is equal to the base, then the exponent is one. For example $\log_4 4 = 1$. If the answer is one, then the exponent is zero. For example, $\log_6 1 = 0$.

The following properties of logarithms can come in handy from time to time.

$\log_x (AB) = (\log_x A) + (\log_x B)$	(Add the logarithms together.)	(0.14)
$\log_x (\frac{A}{B}) = (\log_x A) - (\log_x B)$	(Subtract one logarithm from the other.)	(0.15)
$\log_x (A^N) = N (\log_x A)$	(Multiply the logarithm and the exponent.)	(0.16)
$\log_x A = \frac{\log_b A}{\log_b x}$	(Used to change from one base to another.)	(0.17)

Now You Try It Exercise 0.11: What is $\log_{10}125 = x$ in exponential form?

(A) $10^{125} = x$ (B) $10^{x} = 125$ (C) $x^{10} = 125$ (D) $125^{x} = 10$ (E) $x^{125} = 10$

0.12 Differential Calculus

What is calculus? Often the simplest questions are the hardest to answer. **Calculus** is a branch of mathematics which uses operators to extract information from functions.

Branch of Mathematics	Mathematical Elements Used
Arithmetic	Numbers (0, 1, 101.7, π, e, ...)
Algebra	Variables (x, y, z, t, L, A, ...)
Analysis	Functions ($f(x)$, $v(t)$, $a(t)$, ...)
Calculus	Operators (derivative, integral, Laplacian, ...)

There is a very large number of operators, but for introductory physics only the derivative and integral are needed. There is also a very large number of functions, but the only ones typically needed are constant, polynomial, trigonometric, exponential and logarithmic. These can also be combined by addition, subtraction, multiplication, division and/or composition.

The **derivative** of a function is defined in terms of a limit.

$$\frac{df(x)}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{0.18}$$

Unless a new function is encountered, the limit definition of the derivative is never used. Instead, the limits are determined once for each class of functions, and the resulting expressions, which are called **shortcuts**, are used to calculate them.

Constant	If $f(x) = a$, then $f'(x) = 0$.	(0.19)
Polynomial	If $f(x) = ax^n$, then $f'(x) = nax^{n-1}$.	(0.20)
Trigonometric	If $f(x) = \cos x$, then $f'(x) = -\sin x$ and if $f(x) = \sin x$, then $f'(x) = \cos x$.	(0.21)
Exponential	If $f(x) = e^x$, then $f'(x) = e^x$.	(0.22)
Logarithmic	If $f(x) = \ln x$ where $x > 0$, then $f'(x) = x^{-1}$.	(0.23)

The derivatives of the other four trigonometric functions (tangent, cotangent, secant and cosecant) can be determined by using one or more of the chain rules (see below). Notice the derivative of e^x is itself. This is a very important property of this very special function.

Now You Try It Exercise 0.12: What is the derivative of $f(x) = \frac{5}{x^2}$?

(A) $\frac{10}{x}$ (B) $\frac{-10}{x^3}$ (C) $\frac{5}{2x}$ (D) $\frac{-5}{2x^3}$ (E) $10x$

When the basic types of functions are combined by addition, subtraction, multiplication, division and/or composition, **chain rules** are used to determine the derivatives of the combinations.

Constant Factor	If $f(x) = cg(x)$, then $f'(x) = cg'(x)$.	(0.24)
Addition/Subtraction	If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$.	(0.25)
Multiplication	If $f(x) = g(x)h(x)$, then $f'(x) = g'(x)h(x) + g(x)h'(x)$.	(0.26)
Composition	If $f(x) = g(h(x))$, then $f'(x) = g'(h(x))h'(x)$.	(0.27)

Example 0.14 Determining the Division Chain Rule

If $f(x) = \dfrac{g(x)}{h(x)}$, then $f'(x) = ?$

Write $f(x)$ as the product of two functions: $f(x) = g(x)(h(x))^{-1}$.

Apply the multiplication and composition chain rules: $f'(x) = g'(x)(h(x))^{-1} - g(x)(h(x))^{-2}h'(x)$.

Multiply the first term by $h(x)$ and divide by $h(x)$: $f'(x) = g'(x)(h(x))^{-2}h(x) - g(x)(h(x))^{-2}h'(x)$.

Factor and write as a fraction: $f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$. (0.28)

This is the Division Chain Rule.

Example 0.15 Using the Chain Rules

If $f(x) = 5xe^{2x}$, then $f'(x) = ?$

The derivative of $5x$ is 5 and applying the composition chain rule the derivative of e^{2x} is $2e^{2x}$.

Applying the multiplication chain rule: $f'(x) = (5)(e^{2x}) + (5x)(2e^{2x})$.

Factoring yields: $f'(x) = 5e^{2x}(1 + 2x)$. This is the answer written in its simplest form.

Now You Try It Exercise 0.13: If $f(x) = x^2\sin x$, then $f'(x) = ?$

(A) $2x\sin x$ (B) $x^2\cos x$ (C) $x\sin x(2 + x)$ (D) $x\cos x(2 + x)$ (E) $x(2\sin x + x\cos x)$

0.13 Applications of Differential Calculus

Two applications of the derivative used in physics are determining the slope of the graph of a function at a point and the minimum and/or maximum values of a function.

To determine the **slope** of the graph of a function at a point, calculate the derivative of the function and substitute the value of the variable.

Example 0.16 Slope of a Function at a Point

If $f(x) = 5x^2 + 2x - 3$, then what is the slope at $x = 2$?

Using the shortcut for polynomials combined with the addition chain rule yields $f'(x) = 10x + 2$. The slope at $x = 2$ is then $10(2) + 2 = 22$.

To determine the **minimum** and/or **maximum** values of a function, take its derivative, set it equal to zero, solve for the variable by factoring and using the Zero-Product Rule (see Section 0.2), and substitute the value or values of the variable into the function.

Example 0.17 Maximum Profit

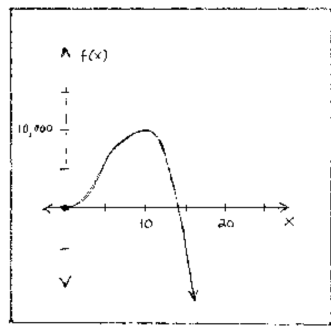
The following function gives the profit (in dollars) for a small company, where f is the profit and x is the number of employees.

$f(x) = 200x^2 - x^4$ where $x > 0$ and x is an integer

Determine the *maximum* profit for the company.

Using the shortcut for polynomials combined with the subtraction chain rule and setting equal to zero yields $400x - 4x^3 = 0$. Factoring yields $4x(10 + x)(10 - x) = 0$. According to the Zero-Product Rule, either $4x = 0$, $(10 + x) = 0$ or $(10 - x) = 0$. Solving gives $x = 0$, -10 or 10. Since the number of employees must be positive, $x = 10$ is the only value of the variable which makes the slope zero. By looking at the graph of the function, one can see that this value of the variable corresponds to the maximum value of the function. (See Figure 0.10.) Substituting 10 into the function yields $f(x) = 200(10)^2 - 10^4 = 10{,}000$ dollars. The maximum profit for the company occurs when there are 10 employees.

Figure



Now You Try It Exercise 0.14: What's the maximum value of $f(x) = 10x(2 - x)$?

(A) 1 (B) 2 (C) 5 (D) 10 (E) 20

<u>0.14 Integral Calculus</u>

The **indefinite integral** of a function can also be defined in terms of a limit, but for our purposes a more useful definition relates it to the derivative. The indefinite integral is the **inverse** of the derivative, and vice versa. That is to say, the derivative of the indefinite integral of a function is equal to the function itself and the indefinite integral of the derivative of a function is equal to the function itself, within a constant called the **constant of integration**.

The symbol for the derivative is *d*, which is an abbreviation for difference, and the symbol for the indefinite integral is $\int$, which looks much like an *s*, which is an abbreviation for sum. Using these symbols, the inverse relationships are:

$$\frac{d}{dx}\int f(x)dx = f(x) \qquad (0.29)$$

$$\int \frac{df(x)}{dx}dx = f(x) + C \qquad (0.30)$$

This pair of equations is called the **Fundamental Theorem of Calculus** and provides an intuitive way of calculating the indefinite integrals of many functions. Since the indefinite integral is the inverse of the derivative, it's often called the **antiderivative**.

Unless a new function is encountered, the limit definition of the indefinite integral is never used. Instead, the limits are determined once for each class of functions, and the resulting expressions, which are called **shortcuts**, are used to calculate them.

Constant — If $f(x) = a$, then $\int f(x)dx = ax + C$. (0.31)
Polynomial — If $f(x) = ax^n$, then $\int f(x)dx = ax^{n+1}/(n + 1) + C$. (0.32)
Trigonometric — $\int \cos x dx = \sin x + C$ and $\int \sin x dx = -\cos x + C$. (0.33)
Exponential — If $f(x) = e^x$, then $\int f(x)dx = e^x + C$. (0.34)
Logarithmic — If $f(x) = \ln x$, then $\int f(x)dx = x(\ln x - 1) + C$. (0.35)

You should verify these shortcuts are correct by applying the derivative to the indefinite integrals of each function. If the shortcut is correct, then the derivative of the indefinite integral will be equal to the original function. Also, notice the indefinite integral of e^x is itself. This is another very important property of this very special function.

Now You Try It Exercise 0.15: What is the indefinite integral of $f(x) = \dfrac{5}{x^2}$?

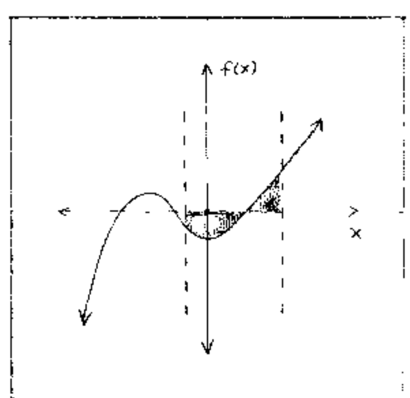
(A) $\dfrac{-10}{x}$ (B) $\dfrac{-10}{x} + C$ (C) $\dfrac{-5}{x}$ (D) $\dfrac{-5}{x} + C$ (E) $10x$ (F) $10x + C$

0.15 Applications of Integral Calculus

Two applications of the integral used in physics are determining area on the graph of a function and the average value of a function.

The **area on the graph of a function** is defined as the area enclosed by the graph of the function, the horizontal axis, and two vertical lines. (See Figure 0.11.)

Figure



If the area is above the horizontal axis, then it's positive and if it's below, then it's negative. If the two vertical lines are located in the same place, then the area is zero.

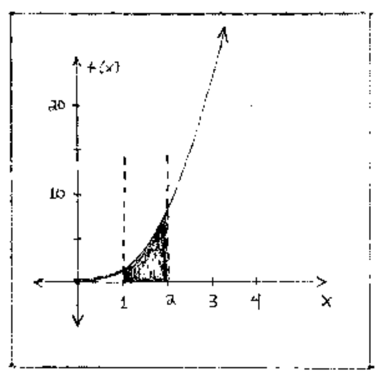
One way to determine this area is to calculate the **definite integral** of the function. This integral is the same as the indefinite integral except it results in a specific value rather than a generalized function. For this kind of integral, the constant of integration is omitted, the two values of the variable defining the vertical lines are substituted and the results are subtracted.

The two values defining the vertical lines are called **limits of integration**. The larger value is called the **upper limit** and the smaller is called the **lower limit**. The result using the lower limit is subtracted from the one using the upper limit. The subset of the domain on the horizontal axis between the two vertical lines is called the **interval**.

Example 0.18 The Area under a Cubic

If $f(x) = x^3$, then what is the area enclosed by the graph of the function, the horizontal axis, and vertical lines at $x = 1$ and $x = 2$? (See Figure 0.12.)

Figure



Using the shortcut for polynomials, the indefinite integral of the function (omitting the constant of integration) is $\frac{x^4}{4}$. Substituting $x = 2$ yields $\frac{2^4}{4} = \frac{16}{4}$ and $x = 1$ yields $\frac{1^4}{4} = \frac{1}{4}$. The area is the difference: $\frac{16}{4} - \frac{1}{4} = \frac{15}{4}$.

Now You Try It Exercise 0.16: What is the area enclosed by the graph of $f(x) = \frac{5}{x^2}$, the horizontal axis, and vertical lines at $x = 1$ and $x = 2$?

(A) 1 (B) $\frac{5}{2}$ (C) 5 (D) $\frac{15}{2}$ (E) 10

To determine the **average value of a function** over some interval, calculate the definite integral over the interval and divide by the length of the interval.

The average value of the function can be thought of as the average height of the graph, measured with respect to the horizontal axis. This method finds the average height by calculating the enclosed area and dividing by the length of the interval. Since area is length multiplied by height, dividing the area by the length of the interval gives the average height.

Example 0.19 The Average Value of a Cubic

Determine the average value of $f(x) = x^3$ over the interval $1 < x < 2$.

Using the shortcut for polynomials, the indefinite integral of the function (omitting the constant of integration) is $\frac{x^4}{4}$. Substituting $x = 2$ yields $\frac{2^4}{4} = \frac{16}{4}$ and $x = 1$ yields $\frac{1^4}{4} = \frac{1}{4}$. The area is the difference: $\frac{16}{4} - \frac{1}{4} = \frac{15}{4}$. The length of the interval is $2 - 1 = 1$. The average value of the function over the interval is then $\frac{\frac{15}{4}}{1} = \frac{15}{4}$.

Now You Try It Exercise 0.17: What is the average value of $f(x) = \frac{5}{x^2}$ over $1 < x < 2$?

(A) 1 (B) $\frac{5}{2}$ (C) 5 (D) $\frac{15}{2}$ (E) 10

Answers to Now You Try It Exercises

(Note: The order of the possible answers can and should be randomized.)

0.1	D
0.2	C
0.3	B
0.4	E
0.5	A
0.6	D
0.7	B
0.8	C
0.9	A
0.10	C
0.11	B
0.12	B
0.13	E
0.14	D
0.15	D
0.16	B
0.17	B

Summary

Algebra includes symbolic manipulation of variables and the use of specialized techniques such as factoring, the Quadratic Formula and the Zero-Product Rule. Geometry includes methods in determining the lengths of curved lines, and the surface areas, cross-sectional areas and volumes of three-dimensional shapes. Trigonometry is the geometry of triangles. The important techniques involve using the Pythagorean Theorem, trigonometric functions (sine, cosine, tangent, secant, cosecant and cotangent), the Law of Cosines and the Law of Sines.

Equations in math are like sentences in English. They are made of symbols, equal signs, numbers and units. Units are arbitrary. Quantities are not. Prefixes are letters that represent numbers and not units. Unit analysis can only tell you if a quantity equation is incorrect. It cannot tell you if it is correct. The number of significant figures is related to the precision of a measurement or given value. The first significant figure is the one that is furthest to the left that is not zero.

There are three ways to solve equations simultaneously. Substitution is the most general method. It always works. Multiply and Subtract only works in certain cases, i.e. those cases where some cancellation will occur when the two equations are subtracted. Forming a ratio only works if there are two equations that involve only multiplication and/or division.

A scalar is simply a number. A vector is comprised of two things: a number and a direction. To add two vectors graphically, use the tail-to-tip method. There are two ways to describe a vector. Polar form gives the magnitude and the direction. Cartesian form is used to add and subtract vectors and it describes how much the vector points horizontally and vertically. These two aspects of a vector are called components. Components that are to the right or up are positive, while those to the left or down are negative. To add two vectors algebraically, use the component method.

An exponential equation is a compact way of writing a number multiplied by itself any number of times. A logarithmic equation is another way of expressing the same information contained in an exponential equation. Technically, exponential and logarithmic functions are inverse functions of each other.

Calculus is a branch of mathematics which uses operators to extract information from functions. The derivative of a function is defined in terms of a limit, but in practice shortcuts are used to calculate the derivatives of the basic types of functions. When the basic types of functions are combined by addition, subtraction, multiplication, division and/or composition, chain rules are used to determine the derivatives of the combinations.

To determine the slope of the graph of a function at a point, calculate the derivative of the function and substitute the value of the variable. To determine the minimum and/or maximum values of a function, take its derivative, set it equal to zero, solve for the variable by factoring and using the Zero-Product Rule, and substitute the value or values of the variable into the function.

The derivative of the indefinite integral of a function is equal to the function itself and the indefinite integral of the derivative of a function is equal to the function itself, within a constant called the constant of integration. The Fundamental Theorem of Calculus provides an intuitive way of calculating the indefinite integrals of many functions. Since the indefinite integral is the inverse of the derivative, it's often called the antiderivative. In practice shortcuts are used to calculate the indefinite integrals of the basic types of functions.

The area on the graph of a function is defined as the area enclosed by the graph of the function, the horizontal axis, and two vertical lines. One way to determine this area is to calculate the definite integral of the function. The two values defining the vertical lines are called limits of integration. The larger value is called the upper limit and the smaller is called the lower limit. The subset of the domain on the horizontal axis between the two vertical lines is called the interval. To determine the average value of a function over some interval, calculate the definite integral over the interval and divide by the length of the interval.

Questions

0.1 From what you have learned by reading this chapter, explain what is wrong with a freeway sign that reads "Speed Limit 40".

0.2 If $\vec{A}$ = (44.0 km, east), $\vec{B}$ = (88.0 km, 30° west of north) and $\vec{C}$ = (76.2 m, north), then which equation is correct? (A) $\vec{B} - \vec{C} = \vec{A}$ (B) $\vec{B} + \vec{C} = \vec{A}$ (C) $\vec{A} + \vec{B} = \vec{C}$ (D) $\vec{A} + \vec{B} + \vec{C} = 0$ (E) $2\vec{A} = \vec{B}$.

0.3 Imagine a right triangle with sides of lengths 4.00 cm and 5.00 cm and a hypotenuse of length 3.00 cm. (A) Is this triangle possible? Explain. (B) If it isn't, then could you reassign the lengths to make a triangle that is possible?

0.4 What is the largest rounding error percentage that could occur with 3 significant figures?

0.5 When can the Quadratic Formula be used? When can't it be used?

0.6 In which situations can the arc length formula be used?

0.7 Compare and contrast surface area and cross-sectional area.

0.8 For what angle does the Law of Cosines reduce to the Pythagorean Theorem?

0.9 What are the MKS units of temperature?

0.10 In practice, are limits used to calculate the derivatives and indefinite integrals of the basic types of functions? If not, then how are they determined?

0.11 Why are chain rules used?

0.12 Why is the indefinite integral often called the antiderivative?

Problems

0.1 (I) If $10x^2 - 100x = -160$, then what are all of the values of x?

0.2 (I) If $|x^2 - 9| = 16$, then what are all of the values of x?

0.3 (I) Convert $x = 0.000000353$ to Scientific Notation.

0.4 (II) A square and an equilateral triangle have the same area. What is the ratio of the length of one side of the square to the length of one side of the triangle?

0.5 (II) If $\sin x = \sin 2x$, then what are all of the values of x in the range $0 \leq x < 360$ degrees? (Hint: Use the double angle identity for sine.)

0.6 (II) The engines of an airplane push it northwest at 100 m/sec while the wind pushes it southwest at 20.0 m/sec. (A) How fast does the airplane move due to the combination of its engines and the wind? (B) In which direction does the airplane move?

0.7 (III) The daily profit (P) equals the number of units sold per day (N) multiplied by the profit per unit. The profit per unit equals the unit price (U) minus the cost of production (C). The number of units sold per day (N) equals some constant (X) determined by experiment minus the unit price (U). The daily profit (P) is \$100, the cost of production (C) is \$2 and the experimental constant (X) is \$104. (A) What is the unit price (U)? Round your answer to the nearest cent. (B) What is the number of units sold per day (N)? Round your answer to the nearest integer.

0.8 (III) Three vectors are related as follows: $\vec{A} + \vec{B} = \vec{C}$. If $\vec{A}$ = (10.0 m, 60.0 degrees north of east) and $\vec{C}$ = (17.3 m, due north), then what is $\vec{B}$?

0.9 (III) If $10 \cos x = 2x$ and $25 \sin x = 5x$, then what are all of the values of x in the range $0 \leq x < 360$ degrees?

0.10 (II) What is the derivative of $f(x) = \sin(x^2)$?

0.11 (II) What is the maximum value of $f(x) = xe^{-x}$?

0.12 (II) What is the average value of $f(x) = e^x$ over the interval $0 < x < 1$?